

ON NON-ARCHIMEDEAN GURARIĬ SPACES

J. KAŁOL, W. KUBIŚ, AND A. KUBZDELA

ABSTRACT. Let \mathcal{U}_{FNA} be the class of all non-archimedean finite-dimensional Banach spaces. A non-archimedean GurariĬ Banach space \mathbb{G} over a non-archimedean valued field \mathbb{K} is constructed, i.e. a non-archimedean Banach space \mathbb{G} of countable type which is of *almost universal disposition* for the class \mathcal{U}_{FNA} . This means: for every isometry $g : X \rightarrow Y$, where $Y \in \mathcal{U}_{FNA}$ and X is a subspace of \mathbb{G} , and every $\varepsilon \in (0, 1)$ there exists an ε -isometry $f : Y \rightarrow \mathbb{G}$ such that $f(g(x)) = x$ for all $x \in X$. We show that all non-archimedean Banach spaces of almost universal disposition for the class \mathcal{U}_{FNA} are ε -isometric. Furthermore, all non-archimedean Banach spaces of almost universal disposition for the class \mathcal{U}_{FNA} are isometrically isomorphic if and only if \mathbb{K} is spherically complete and $\{|\lambda| : \lambda \in \mathbb{K} \setminus \{0\}\} = (0, \infty)$.

1. INTRODUCTION

In 1966 GurariĬ constructed a separable (real) Banach space \mathbb{G} of *almost universal disposition* for finite-dimensional spaces (called later the *GurariĬ space*), see [3], which means the following condition:

- (G) *For every isometry $g : X \rightarrow Y$, where Y is a finite-dimensional Banach space and X is a subspace of \mathbb{G} , and every $\varepsilon \in (0, 1)$ there exists an ε -isometry $f : Y \rightarrow \mathbb{G}$ such that $f(g(x)) = x$ for all $x \in X$.*

A linear operator $f : E \rightarrow F$ between Banach spaces E and F is an ε -isometry if for $x \in E$ with $\|x\| = 1$ one has $(1 + \varepsilon)^{-1} < \|f(x)\| < 1 + \varepsilon$. By an *isometry* we mean a linear operator $f : E \rightarrow F$ that is an ε -isometry for every $\varepsilon > 0$, that is, $\|f(x)\| = \|x\|$ for each $x \in E$.

One can prove easily that the GurariĬ space \mathbb{G} is unique up to isomorphism of norm arbitrarily close to one. Nevertheless, the question whether the GurariĬ space is unique up to isometry remained open for a longer time. It was answered affirmatively by Lusky in 1976, see [10], who used quite technical and difficult methods involving techniques developed by Lazar and Lindenstrauss [9]. Much simpler proof has been provided by Kubiś and Solecki in 2013, see [5].

In [5] the authors proved the following

Theorem 1.1. *Let E, F be separable GurariĬ spaces and $\varepsilon > 0$. Assume $X \subset E$ is a finite-dimensional space and $f : X \rightarrow F$ is an ε -isometry. Then there exists a bijective isometry $h : E \rightarrow F$ such that $\|h|_X - f\| < \varepsilon$.*

2010 *Mathematics Subject Classification.* 46S10.

Key words and phrases. Non-archimedean Banach spaces, Isometric embedding, Universal disposition.

The first named author was supported by Generalitat Valenciana, Conselleria d'Educació, Cultura i Esport, Spain, Grant PROMETEO/2015/058 and by the GAČR project 16-34860L and RVO: 67985840.

The second author was supported by the GAČR project 16-34860L and RVO: 67985840.

Applying this result for X being the trivial space one gets the result of Lusky [10] stating that the Gurarii space is unique up to isometry.

A Banach space E is said to be of *universal disposition* for the class \mathcal{U} of finite-dimensional spaces if it satisfies the following condition:

- (G1) *For every isometry $j : X \rightarrow Y$, where $Y \in \mathcal{U}$ and $X \subset E$, there is an isometry $f : Y \rightarrow E$ such that $f(j(x)) = x$ for all $x \in X$.*

We refer the reader to [1] and [2], where recent developments in the study of Gurarii spaces, spaces of universal disposition, and related topics are surveyed.

In the present paper we study non-archimedean counterparts of the above concepts. The property of being of (almost) universal disposition for finite-dimensional non-archimedean normed spaces is defined precisely in the same way as for the real case mentioned above.

From now on, by \mathbb{K} we will denote a non-archimedean complete non-trivially valued field, i.e. the valuation satisfies *the strong triangle inequality*:

$$|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\}$$

for all $\lambda, \mu \in \mathbb{K}$.

All linear spaces considered in this paper are over \mathbb{K} . Recall that

$$|\mathbb{K}^*| = \{|\lambda| : \lambda \in \mathbb{K} \setminus \{0\}\}$$

is the *value group* of \mathbb{K} .

\mathbb{K} is said to be *discretely valued* if 0 is the only accumulation point of $|\mathbb{K}^*|$; then, there exists a *uniformizing element* $\rho \in \mathbb{K}$ with $0 < |\rho| < 1$ such that $|\mathbb{K}^*| = \{|\rho|^n : n \in \mathbb{Z}\}$. Otherwise, we say that \mathbb{K} is *densely valued* (then, $|\mathbb{K}^*|$ is a dense subset of $[0, \infty)$).

By a *non-archimedean Banach space* we mean a Banach space E equipped with a non-archimedean norm $\|\cdot\|$, i.e. a norm for which the triangle inequality is replaced by a stronger condition $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in E$.

An infinite-dimensional normed space E over \mathbb{K} is of *countable type* if it contains a countable set whose linear hull is dense in E . If \mathbb{K} is separable, then a normed space is of countable type if and only if it is separable.

We say that E (in particular, E may be equal to \mathbb{K}) is *spherically complete* if every shrinking sequence of balls in E has a non-empty intersection; otherwise, E is *non-spherically complete*. Every finite-dimensional Banach space over \mathbb{K} has an equivalent non-archimedean norm. We refer the reader to the monographs [11] and [12] for non-archimedean concepts mentioned above.

We say that a spherically complete Banach space \widehat{E} is the *spherical completion* of a non-archimedean Banach space E , if there exists an isometric embedding $i : E \rightarrow \widehat{E}$ and \widehat{E} has no proper spherically complete linear subspace containing $i(E)$. Applying the natural identification, we will usually identify E with $i(E)$. Every Banach space (in particular \mathbb{K}) has the spherical completion and any two spherical completions of E are isometrically isomorphic ([12, Theorem 4.43]).

Let \mathcal{U}_{FNA} be the class of all non-archimedean finite-dimensional normed spaces. As it can be expected, properties of spaces E of (almost) universal disposition for the class \mathcal{U}_{FNA} strictly depend on the valued field \mathbb{K} , in particular, on whether it is spherically complete or not. In Section 3 we show that all non-archimedean Banach spaces of almost universal disposition for the class \mathcal{U}_{FNA} are ε -isometric with arbitrarily small $\varepsilon > 0$ (Corollary 3.3). Moreover, all non-archimedean Banach spaces of almost universal disposition for

the class \mathcal{U}_{FNA} are isometrically isomorphic if and only if \mathbb{K} is spherically complete and $|\mathbb{K}^*| = (0, \infty)$ (Proposition 3.4). The main result of this section is the following

Theorem 1.2. *Let \mathbb{K} be a non-archimedean valued field. The following conditions are equivalent:*

- (a) *Every non-archimedean Banach space of countable type over \mathbb{K} is of almost universal disposition for the class \mathcal{U}_{FNA} .*
- (b) *\mathbb{K} is densely valued.*

Section 4 focuses on the study Banach spaces of universal disposition for the class \mathcal{U}_{FNA} . It turns out that a real Banach space G of almost universal disposition for the class \mathcal{U} can be characterized (see [4]) by the following condition:

- (H) for every $\varepsilon > 0$, for every finite-dimensional normed spaces $X \subset Y$ and for every isometric embedding $j : X \rightarrow G$, there is an isometric embedding $f : Y \rightarrow G$ such that $\|j - f|_X\| < \varepsilon$.

In contrast to the real case, in a non-archimedean setting the condition (H) characterizes Banach spaces of universal disposition for the class \mathcal{U}_{FNA} . We show the following

Theorem 1.3. *Let G be a non-archimedean Banach space which satisfies the following property: For every finite-dimensional non-archimedean normed space Y and every isometric embedding $j : X \rightarrow G$, where $X \subset Y$ is a linear subspace, there is an isometry $f : Y \rightarrow G$ such that $\|j - f|_X\| < 1$. Then G is of universal disposition for the class \mathcal{U}_{FNA} .*

Recall that a Banach space X is (*isometrically*) *universal* for the class of Banach spaces \mathcal{C} if $X \in \mathcal{C}$ and for any $Y \in \mathcal{C}$, there is an isometrical embedding $Y \rightarrow X$. Note that, as a result of Banach-Mazur theorem, the space $C[0, 1]$ is isometrically universal for the class of separable real Banach spaces.

If \mathbb{K} is spherically complete, we can properly select a set I and a map $s : I \rightarrow (0, \infty)$ such that every non-archimedean Banach space of countable type can be isometrically embedded into $E_u = c_0(I : s)$. However E_u is isometrically universal for the class of non-archimedean Banach spaces of countable type if and only if \mathbb{K} is spherically complete and $(0, \infty)$ is an union of at most countably many cosets of $|\mathbb{K}^*|$ (Proposition 4.4). On the other hand, E_u is never separable. If \mathbb{K} is non-spherically complete, the role of $c_0(I : s)$ is replaced by ℓ^∞ , which clearly is not of countable type (Remark 4.5).

Applying Theorem 1.3 we prove that the spherical completion \widehat{E}_u of E_u is a space of universal disposition for the class \mathcal{U}_{FNA} , see Theorem 4.6. We show also that the suitably selected proper linear subspace of \widehat{E}_u , denoted as E_h , is also of universal disposition for the class \mathcal{U}_{FNA} (Theorem 4.7). If \mathbb{K} is spherically complete, then $E_h = E_u$; hence, E_h has an orthogonal base and is of countable type if and only if \mathbb{K} is spherically complete and $(0, \infty)$ is the union of at most countably many cosets of $|\mathbb{K}^*|$ (Corollary 4.8).

2. PRELIMINARIES

Let $t \in (0, 1]$. A subset $\{x_i : i \in I\} \subset E$ is called *t-orthogonal* (*orthogonal* for $t = 1$) if for each finite subset $J \subset I$ and all $\{\lambda_i\}_{i \in J} \subset \mathbb{K}$ we have

$$\left\| \sum_{i \in J} \lambda_i x_i \right\| \geq t \cdot \max_{i \in J} \|\lambda_i x_i\|.$$

If additionally $\overline{\{x_i\}_{i \in I}} = E$, then $\{x_i\}_{i \in I}$ is said to be a t -orthogonal base of E . Then every $x \in E$ has an unequivocal expansion

$$x = \sum_{i \in I} \lambda_i x_i \quad (\lambda_i \in \mathbb{K}, \quad i \in I).$$

Every non-archimedean Banach space of countable type has a t -orthogonal base for each $t \in (0, 1)$; if \mathbb{K} is spherically complete, then every non-archimedean Banach space of countable type has an orthogonal base ([12, Lemma 5.5]). Every closed linear subspace of a non-archimedean Banach space with an orthogonal base has an orthogonal base ([12, Theorem 5.9]).

Linear subspaces D, D_0 of a non-archimedean Banach space E are called *orthogonal* if $\|x + y\| = \max\{\|x\|, \|y\|\}$ for all $x \in D$ and $y \in D_0$; then we will write $D \perp D_0$.

Let D be a closed linear subspace of E . Then D is *orthocomplemented* in E if there is a linear subspace D_0 of E such that $D + D_0 = E$ and $D \perp D_0$. Consequently, there exists a surjective projection (called an *orthoprojection*) $P : E \rightarrow D$ with $\|P\| \leq 1$. Observe that $D_1 \perp D_2$ implies $D_1 \cap D_2 = \emptyset$; hence the sum $D_1 + D_2$ is direct.

Let D and E_0 be linear subspaces of a normed space E . Recall that E_0 is called an *immediate extension* of D if $D \subset E_0$ and there is no nonzero element of E_0 that is orthogonal to D ; in other words, for every $x \in E_0 \setminus D$ we have $\text{dist}(x, D) < \|x - d\|$ for all $d \in D$. A spherical completion \widehat{E} of E is a maximal immediate extension of E . Let I be a non-empty set and let $s : I \rightarrow (0, \infty)$ be a map. By

$$\ell^\infty(I : s) := \{(\lambda^i)_{i \in I} \in \mathbb{K}^I : \sup_{i \in I} |\lambda_i| \cdot s(i) < \infty\}$$

we denote the linear space over \mathbb{K} equipped with the norm

$$\|(\lambda_i)_{i \in I}\| := \sup_{i \in I} |\lambda_i| \cdot s(i).$$

Then $\ell^\infty(I : s)$ is a non-archimedean Banach space.

Let $c_0(I : s)$ be a closed linear subspace of $\ell^\infty(I : s)$ which consists of all $(\lambda^i)_{i \in I} \in \ell^\infty(I : s)$ such that for every $\varepsilon > 0$ there exists a finite $J \subset I$ for which $|\lambda^i| \cdot s(i) < \varepsilon$ for every $i \in I \setminus J$. If $s(i) = 1$ for all $i \in I$ we will write $\ell^\infty(I)$ and $c_0(I)$, respectively. In particular $\ell^\infty := \ell^\infty(\mathbb{N})$ and $c_0 := c_0(\mathbb{N})$.

Every non-archimedean Banach space which has an orthogonal base is isomorphic with $c_0(I)$ for some set I (see [12, Ch. 5]).

3. NON-ARCHIMEDEAN BANACH SPACE OF ALMOST UNIVERSAL DISPOSITION FOR FINITE-DIMENSIONAL SPACES

First we prove the following technical fact.

Lemma 3.1. *Let E be a non-archimedean Banach space of countable type, let F be a finite-dimensional linear subspace of E , $t \in (0, 1)$ and $\{x_1, \dots, x_m\}$ be a \sqrt{t} -orthogonal base of F . Then there exist $x_{m+1}, x_{m+2}, \dots \in E \setminus F$ such that (x_n) is a t -orthogonal base of E .*

Proof. By [11, Theorem 2.3.13] there exists a linear subspace $F_0 \subset E$ such that $E = F \oplus F_0$ and

$$\|u_1 + u_2\| \geq \sqrt{t} \cdot \max\{\|u_1\|, \|u_2\|\}$$

for all $u_1 \in F, u_2 \in F_0$. Applying [11, Theorem 2.3.7] we select a \sqrt{t} -orthogonal base (z_n) of F_0 . Denote $x_{m+n} := z_n$ for every $n \in \mathbb{N}$. Then taking any $k \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_k \in \mathbb{K}$ one gets

$$\begin{aligned} \left\| \sum_{i=1}^k \lambda_i x_i \right\| &\geq \sqrt{t} \cdot \max \left\{ \left\| \sum_{i=1}^m \lambda_i x_i \right\|, \left\| \sum_{i=m+1}^k \lambda_i x_i \right\| \right\} \\ &\geq \sqrt{t} \cdot \max \left\{ \sqrt{t} \cdot \max_{i=1, \dots, m} \|\lambda_i x_i\|, \sqrt{t} \cdot \max_{i=m+1, \dots, k} \|\lambda_i x_i\| \right\} \geq t \cdot \max_{i=1, \dots, k} \|\lambda_i x_i\|. \end{aligned}$$

Hence, (x_n) is a required t -orthogonal base of E . \square

Theorem 3.2. *A non-archimedean Banach space of countable type E is of almost universal disposition for the class \mathcal{U}_{FNA} if and only if $\|E^*\|$ is a dense subset of $(0, \infty)$.*

Proof. Let E be a non-archimedean Banach space of countable type. Assume that $\|E^*\|$ is a dense subset of $(0, \infty)$. Let X be a finite-dimensional subspace of E , Y be a non-archimedean finite-dimensional normed space and $i : X \rightarrow Y$ be an isometrical embedding. Assume that $\dim Y = m$ and $\dim X = m_0$; clearly, $m_0 \leq m$. Fix $\varepsilon > 0$ and take $t \in (\frac{1}{\sqrt[3]{1+\varepsilon}}, 1)$. Applying Lemma 3.1 and [11, Theorem 2.3.7] we form a t -orthogonal base (x_n) of E such that $\{x_1, \dots, x_{m_0}\}$ is a \sqrt{t} -orthogonal base of X . Now, applying Lemma 3.1 again, we select $y_{m_0+1}, \dots, y_m \in Y$ such that $\{y_1, \dots, y_m\}$ is a t -orthogonal base of Y , where $y_k = i(x_k)$ for $k \in \{1, \dots, m_0\}$. Since, by assumption $\|E^*\|$ is a dense subset of $(0, \infty)$, we can assume that $1 \geq \|y_k\| \geq t$ ($k = 1, \dots, m$) and $1 \geq \|x_n\| \geq t$ ($n \in \mathbb{N}$); thus, $\|y_k\| \geq t \cdot \|x_k\|$ for each $k \in \{1, \dots, m\}$.

Define $f : Y \rightarrow E$ by setting

$$f \left(\sum_{k=1}^m \lambda_k y_k \right) = \sum_{k=1}^m \lambda_k x_k.$$

Clearly, $f(i(x)) = x$ for all $x \in X$. Let $y = \sum_{k=1}^m \lambda_k y_k \in Y$. Then,

$$\|y\| = \left\| \sum_{k=1}^m \lambda_k y_k \right\| \geq t \cdot \max_{k=1, \dots, m} \{\|\lambda_k y_k\|\} \geq t^2 \cdot \max_{k=1, \dots, m} \{\|\lambda_k x_k\|\} \geq t^3 \cdot \left\| \sum_{k=1}^m \lambda_k x_k \right\| = t^3 \cdot \|f(y)\|.$$

On the other hand, we have

$$\|y\| = \left\| \sum_{k=1}^m \lambda_k y_k \right\| \leq \max_{k=1, \dots, m} \{\|\lambda_k y_k\|\} \leq \frac{1}{t} \max_{k=1, \dots, m} \{\|\lambda_k x_k\|\} \leq \frac{1}{t^2} \left\| \sum_{k=1}^m \lambda_k x_k \right\| = \frac{1}{t^2} \cdot \|f(y)\|.$$

Thus $(1 - \varepsilon) \|y\| \leq \|f(y)\| \leq (1 + \varepsilon) \|y\|$.

Now assume that $\|E^*\|$ is not dense in $(0, \infty)$. Then there exist $s_1 \in (0, \infty)$ and $\varepsilon > 0$ such that

$$\|E^*\| \cap (s_1 - 2\varepsilon \cdot s_1, s_1 + 2\varepsilon \cdot s_1) = \emptyset.$$

Define $X = \mathbb{K}$ and $Y = (\mathbb{K}^2, \|\cdot\|_Y)$, where $\|(\lambda_1, \lambda_2)\|_Y := \max\{|\lambda_1|, s_1 \cdot |\lambda_2|\}$, $\lambda_1, \lambda_2 \in \mathbb{K}$.

Assume for a contradiction that there is an ε -isometry $f : Y \rightarrow E$. But then, taking $x_0 = (0, 1) \in Y$, we obtain $\|x_0\|_Y = s_1$ and $\|f(x_0)\| \geq s_1 + 2\varepsilon \cdot s_1$ or $\|f(x_0)\| \leq s_1 - 2\varepsilon \cdot s_1$. Hence, $\|f(x_0)\| > (1 + \varepsilon) \|x_0\|$, or $\|f(x_0)\| < (1 - \varepsilon) \|x_0\|$, a contradiction. \square

Next conclusion follows directly from Theorem 3.2.

Corollary 3.3. *If \mathbb{K} is densely valued, every non-archimedean Banach space of countable type is of almost universal disposition for the class \mathcal{U}_{FNA} . All non-archimedean Banach spaces of almost universal disposition for the class \mathcal{U}_{FNA} are ε -isometric, where $\varepsilon > 0$ is arbitrarily small.*

However, as the next result shows, they need not be always isometric.

Proposition 3.4. *All non-archimedean Banach spaces over \mathbb{K} of almost universal disposition for the class \mathcal{U}_{FNA} are isometrically isomorphic if and only if \mathbb{K} is spherically complete and $|\mathbb{K}^*| = (0, \infty)$.*

Proof. If \mathbb{K} is spherically complete and $|\mathbb{K}^*| = (0, \infty)$ then every non-archimedean Banach space of countable type has an orthonormal base, thus, it is isometrically isomorphic with c_0 . Hence, the conclusion follows.

Now assume that \mathbb{K} is non-spherically complete. Then, since \mathbb{K} is densely valued, c_0 and $\mathbb{K}_v^2 \oplus c_0$ are both of universal disposition for the class \mathcal{U}_{FNA} by Theorem 3.2 (recall that \mathbb{K}_v^2 is a two-dimensional normed space without two orthogonal elements, see [11, Example 2.3.26]). Clearly, $\mathbb{K}_v^2 \oplus c_0$ and c_0 are not isometrically isomorphic.

Suppose that $|\mathbb{K}^*| \neq (0, \infty)$. Then we can find $s \in (0, \infty) \setminus |\mathbb{K}^*|$. Define the norm $\|x\|_s : c_0 \rightarrow [0, \infty)$ by

$$\|x\|_s := \max\{s \cdot |x_1|, \max_{n>1}\{|x_n|\}\},$$

and $x = (x_n) \in c_0$. Then, by Theorem 3.2, $E = (c_0, \|\cdot\|_s)$ and $F = (c_0, \|\cdot\|_\infty)$ are of almost universal disposition for the class \mathcal{U}_{FNA} . Since $\|E\| \neq \|F\|$, E and F are not isometrically isomorphic. \square

Now, we are ready to prove Theorem 1.2, which characterizes \mathbb{K} , formulated in Introduction.

Proof of Theorem 1.2. Let E be a non-archimedean Banach space of countable type. If \mathbb{K} is densely valued, $\|E^*\|$ is a dense subset of $(0, \infty)$ and the conclusion follows from Theorem 3.2. Assume now that \mathbb{K} is discretely valued and ρ be a uniformizing element of \mathbb{K} . Let $E := c_0$. Set $s := \frac{|\rho|+1}{2}$ and take

$$\varepsilon < \frac{1-s}{s} = \frac{s-|\rho|}{s}.$$

Let $X = [e_1] \subset E$ and $Y = (\mathbb{K}^2, \|\cdot\|_s)$, where

$$\|x\|_s := \max\{|x_1|, s \cdot |x_2|\}, (x_1, x_2) \in \mathbb{K}^2.$$

Define an isometry $i : X \rightarrow Y$ by $i(\lambda e_1) := (\lambda x_1, 0)$ and assume that there exists an ε -isometry $f : Y \rightarrow E$. Then, for $x = (0, 1) \in Y$ we get

$$(1 - \varepsilon) \cdot s \leq \|f(x)\| \leq (1 + \varepsilon) \cdot s.$$

Recall that $\|E^*\| = \{|\rho|^n : n \in \mathbb{Z}\}$, hence $(|\rho|, 1) \cap \|E^*\| = \emptyset$. But $(1 - \varepsilon) \cdot s > |\rho|$ and $(1 + \varepsilon) \cdot s < 1$, a contradiction. \square

4. NON-ARCHIMEDEAN BANACH SPACE OF UNIVERSAL DISPOSITION FOR FINITE-DIMENSIONAL SPACES

We start with the proof of Theorem 1.3, as promised in Introduction.

Proof of Theorem 1.3. We need to prove that there is an isometry $T : Y \rightarrow G$ which extends j . Let $t = \|j - f|_X\|$, $n_X = \dim X$ and $n_Y = \dim Y$. Clearly $n_X \leq n_Y$. Choose $\{x_1, \dots, x_{n_Y}\}$, a \sqrt{t} -orthogonal base of Y such that $[x_1, \dots, x_{n_X}] = X$. Define $T : Y \rightarrow G$ by setting

$$T(x_n) := \begin{cases} j(x_n) & \text{if } n \leq n_X \\ f(x_n) & \text{if } n > n_X \end{cases}, n = 1, \dots, n_Y.$$

Clearly T extends j . We show that T is an isometry. Take $x \in Y$, written as $x = \sum_{i=1}^{n_Y} \lambda_i x_i$ ($\lambda_i \in \mathbb{K}$). Then, we obtain

$$\begin{aligned} \|T(x)\| &= \left\| \sum_{i=1}^{n_X} \lambda_i T(x_i) + \sum_{i=n_X+1}^{n_Y} \lambda_i T(x_i) \right\| \\ &= \left\| \sum_{i=1}^{n_X} \lambda_i j(x_i) - \sum_{i=1}^{n_X} \lambda_i f(x_i) + \sum_{i=1}^{n_X} \lambda_i f(x_i) + \sum_{i=n_X+1}^{n_Y} \lambda_i f(x_i) \right\| \\ &= \left\| (j - f)\left(\sum_{i=1}^{n_X} \lambda_i x_i\right) + f\left(\sum_{i=1}^{n_Y} \lambda_i x_i\right) \right\| = \|(j - f)(x_0) + f(x)\|, \end{aligned}$$

where $x_0 = \sum_{i=1}^{n_X} \lambda_i x_i \in X$. But,

$$\|x\| \geq \sqrt{t} \cdot \max \left\{ \|x_0\|, \sum_{i=n_X+1}^{n_Y} \lambda_i x_i \right\} \geq \sqrt{t} \cdot \|x_0\|$$

and

$$\|(j - f)(x_0)\| \leq \|j - f|_X\| \cdot \|x_0\| = t \cdot \|x_0\| \leq \sqrt{t} \cdot \|x\| < \|x\| = \|f(x)\|$$

hence, we get

$$\|T(x)\| = \|(j - f)(x_0) + f(x)\| = \|f(x)\| = \|x\|.$$

□

To prove the main result of this section we need a few technical lemmas (see also [12, Exercise 5.B and Lemma 4.42] and [11, Theorem 2.3.16]).

Lemma 4.1. *Let $0 < t \leq 1$ and let $\{x_1, \dots, x_n\}$ be a t -orthogonal set in a non-archimedean normed space E . If $\{z_1, \dots, z_n\} \subset E$ and $\|x_i - z_i\| < t \cdot \|x_i\|$ for each $i \in \{1, \dots, n\}$, then $\{z_1, \dots, z_n\}$ is also t -orthogonal.*

Proof. Take any $\lambda_i \in \mathbb{K}$, $i = 1, \dots, n$. Since $\|x_i - z_i\| < t \cdot \|x_i\|$, we have $\|z_i\| = \|x_i\|$ for each $i \in \{1, \dots, n\}$. Consequently we note

$$\|\lambda_1 x_1 + \dots + \lambda_n x_n\| \geq t \cdot \max_{i=1, \dots, n} \|\lambda_i x_i\|$$

and

$$\|\lambda_1(z_1 - x_1) + \dots + \lambda_n(z_n - x_n)\| \leq \max_{i=1, \dots, n} \|\lambda_i(z_i - x_i)\| < t \cdot \max_{i=1, \dots, n} \|\lambda_i x_i\|.$$

Hence

$$\begin{aligned} \|\lambda_1 z_1 + \dots + \lambda_n z_n\| &= \|\lambda_1 z_1 + \dots + \lambda_n z_n - (\lambda_1 x_1 + \dots + \lambda_n x_n) + (\lambda_1 x_1 + \dots + \lambda_n x_n)\| \\ &= \|\lambda_1(z_1 - x_1) + \dots + \lambda_n(z_n - x_n) + (\lambda_1 x_1 + \dots + \lambda_n x_n)\| \\ &= \|\lambda_1 x_1 + \dots + \lambda_n x_n\| \geq t \cdot \max_{i=1, \dots, n} \|\lambda_i x_i\| = t \cdot \max_{i=1, \dots, n} \|\lambda_i z_i\|, \end{aligned}$$

and we are done. \square

Lemma 4.2. *Let E, F be non-archimedean normed spaces, D a linear subspace of E such that E is an immediate extension of D , let F be spherically complete and $T : D \rightarrow F$ be an isometry. Then T can be extended to a linear isometry $T' : E \rightarrow F$.*

Proof. Applying Ingleton's theorem (see [12, Theorem 4.8]) we can extend the isometry $T : D \rightarrow F$ to the linear operator $T' : E \rightarrow F$ such that $\|T'\| \leq 1$. We prove that T' is also an isometry.

Set $x \in E \setminus D$. Then, since E is an immediate extension of D , there is $x_d \in D$ such that

$$\|x - x_d\| < \|x_d\| = \|x\|.$$

Thus we get

$$\|T'(x) - T'(x_d)\| \leq \|T'\| \cdot \|x - x_d\| < \|x_d\| = \|T'(x_d)\|$$

and

$$\|T'(x)\| = \|T'(x) - T'(x_d) + T'(x_d)\| = \|T'(x_d)\|.$$

Hence, finally $\|T'(x)\| = \|x\|$. \square

Lemma 4.3. *Let Y be a finite-dimensional non-archimedean normed space and X be its linear subspace. Let $\{u_1, \dots, u_{m_Y}\}$ be a maximal orthogonal set in Y such that $\{u_1, \dots, u_{m_X}\}$ is a maximal orthogonal set in X for some $m_X \leq m_Y$ and let $F_Y := [u_{m_X+1}, \dots, u_{m_Y}]$. Then, $F_Y \perp X$.*

Proof. Assume that $m_Y > m_X$ (otherwise nothing is to prove). Take any $x \in X$ and $y \in F_Y$. If $x \in [u_1, \dots, u_{m_X}]$, the conclusion is obvious. So, assume that $x \notin [u_1, \dots, u_{m_X}]$. But then, since X is an immediate extension of $[u_1, \dots, u_{m_X}]$, there is $x_0 \in [u_1, \dots, u_{m_X}]$ with

$$\|x - x_0\| < \|x\| = \|x_0\|.$$

Thus, since

$$\|x_0 + y\| = \max\{\|x_0\|, \|y\|\},$$

we have

$$\|x + y\| = \|x - x_0 + x_0 + y\| = \|x_0 + y\| = \max\{\|x_0\|, \|y\|\} = \max\{\|x\|, \|y\|\}.$$

\square

Let $r := |\rho|$ if \mathbb{K} is discretely valued, where $\rho \in \mathbb{K}$ is a uniformizing element of $|\mathbb{K}^*|$ with $0 < |\rho| < 1$, and let r be any number taken from $(\frac{1}{2}, 1)$ if \mathbb{K} is densely valued. Note that $(0, \infty)$ is a multiplicative group. Let

$$(4.2) \quad \pi : (0, \infty) \rightarrow G := (0, \infty) / |\mathbb{K}^*|$$

be the quotient map and let $S = \{s_g : g \in G\}$ be a set of representatives of elements of G in $(r, 1]$, i.e. $\pi(s_g) = g$.

Let I_u be a set for which $\text{card}(I_u) = \max\{\aleph_0, \text{card}(G)\}$ and let $I_u = \bigcup_{g \in G} I_g$ where $\{I_g : g \in G\}$ is a partition of I_u such that $\text{card}(I_g) = \aleph_0$ for each $g \in G$. Then, clearly $c_0(I_u) = \bigoplus_{g \in G} c_0(I_g)$.

Define the function $s : I_u \rightarrow (r, 1]$ by $h(i) := s_g$ if $i \in I_g$ and the norm on $c_0(I_u)$ by

$$\|x\|_u := \max_{i \in I_u} \{s(i) \cdot |x_i|\}, \quad x = (x_i)_{i \in I} \in c_0(I_u).$$

Denote $E_u := (c_0(I_u), \|\cdot\|_u)$.

Proposition 4.4. *Let \mathbb{K} be spherically complete. Then every non-archimedean Banach space of countable type can be isometrically embedded into E_u .*

- E_u is of countable type (hence E_u is isometrically universal for the class of non-archimedean Banach spaces of countable type) if and only if $(0, \infty)$ is the union of at most countably many cosets of $|\mathbb{K}^*|$;
- E_u is never separable.

Proof. Let E be a non-archimedean Banach space of countable type. Since \mathbb{K} is spherically complete, E has an orthogonal base (x_n) (see [11, Theorem 2.3.25]). Let

$$J_g = \{n : \pi(\|x_n\|) = g\}, \quad g \in G,$$

where π is the map defined in (4.2). Then $G_0 = \{g \in G : J_g \text{ is nonempty}\}$ is at most countable. So, we can write $\mathbb{N} = \bigcup_{g \in G_0} J_g$, where J_g ($g \in G_0$) are nonempty, finite or infinite, pairwise disjoint subsets of \mathbb{N} .

Define the map $l : \mathbb{N} \rightarrow I_u$ (recall that $I_u = \bigcup_{g \in G} I_g$ and I_g is countable for every $g \in G$; thus we can write $I_g = \{m_1^g, m_2^g, \dots\}$, $g \in G$) as follows: for every $n \in \mathbb{N}$ there exist $g \in G_0$ and $k \in \mathbb{N}$ such that $n = n_k^g$ (what means $n \in J_g$). Finally set $l(n) := m_k^g$.

Note that for every $n \in \mathbb{N}$ we can find $\lambda_n \in \mathbb{K}$ for which

$$\|x_n\| = s_g \cdot |\lambda_n|.$$

Next, define the map $i_0 : \{x_1, x_2, \dots\} \rightarrow E_u$ by the formula $x_n \mapsto \lambda_n e_{l(n)}$, where e_n are as usual the unit vectors. Since

$$\|\lambda_n e_{l(n)}\|_u = s_g \cdot |\lambda_n| = \|x_n\|,$$

we can extend the map i_0 to an isometric embedding $E \rightarrow E_u$.

Now we prove the next claim of the proposition. Suppose that $(0, \infty)$ is not the union of at most countably many cosets of $|\mathbb{K}^*|$ and assume that E_u is of countable type. By [11, Theorem 2.3.25] the space E_u has an orthogonal base (x_n) . Since (x_n) is orthogonal, $\|E_u\|_u \setminus \{0\}$ consist of at most countably many cosets of $|\mathbb{K}^*|$. Hence there exists

$$s \in (0, \infty) \setminus \|E_u\|_u.$$

Define $E = (\mathbb{K}^2, \|\cdot\|_s)$, where

$$\|(x, y)\|_s := \max\{|x|, s \cdot |y|\},$$

$(x, y) \in \mathbb{K}^2$. Then

$$\|(0, 1)\|_s \notin \|E_u\|_u.$$

Hence there is no isometry $E \rightarrow E_u$, a contradiction. If $(0, \infty)$ is the union of at most countably many cosets of $|\mathbb{K}^*|$, then G , and consequently I_u is countable, and E_u is of countable type.

Finally, assume that \mathbb{K} is separable. If \mathbb{K} is discretely valued, then $(0, \infty)$ is always the union of more than countably many cosets of $|\mathbb{K}^*|$. On the other hand, by [13, Theorem 20.5] there is no separable densely valued spherically complete \mathbb{K} ; hence E_u is not separable. \square

Remark 4.5. If \mathbb{K} is non-spherically complete, then E_u does not contain an isometric image of any non-archimedean Banach spaces of countable type. Indeed, in this case there exists finite-dimensional normed spaces without orthogonal bases, see [11, Example 2.3.26] and [6]. Take $E = \mathbb{K}_v^2$, where \mathbb{K}_v^2 is a two-dimensional normed space over \mathbb{K} without two non-zero orthogonal elements, and assume that there exists an isometric embedding $i : E \rightarrow E_u$. Then the image $i(E)$ has no two non-zero orthogonal elements. But this contradicts the conclusion of Gruson's theorem ([12, Theorem 5.9]) stating that every linear subspace of a non-archimedean Banach spaces with an orthogonal base has an orthogonal base.

If \mathbb{K} is non-spherically complete the role of $c_0(I : s)$ takes the space ℓ^∞ . In this case, by [11, Theorem 2.5.13], every non-archimedean Banach space of countable type can be isometrically embedded into ℓ^∞ .

Finally we prove the following

Theorem 4.6. *The spherical completion $\widehat{E_u}$ of E_u is a non-archimedean Banach space of universal disposition for the class \mathcal{U}_{FNA} .*

Proof. Denote $F := \widehat{E_u}$. Let $X \subset F$ and let $j : X \rightarrow Y$ be an isometric embedding, where Y is a finite non-archimedean normed space. We prove, that there exists an isometric embedding $f : Y \rightarrow F$ such that $f(j(x)) = x$ for all $x \in X$.

Choose a maximal orthogonal set $\{u_1, \dots, u_{m_X}, \dots, u_{m_Y}\}$ in Y such that $\{u_1, \dots, u_{m_X}\}$ is a maximal orthogonal set in $j(X)$ for some $m_Y \geq m_X \geq 1$. Set $F_Y := [u_{m_X+1}, \dots, u_{m_Y}]$. By Lemma 4.3 we get $F_Y \perp j(X)$.

Set $v_k = f(u_k) := j^{-1}(u_k)$ for each $k \in \{1, \dots, m_X\}$. For every $k \in \{m_X+1, \dots, m_Y\}$ choose $i_k \in I$ such that $\|e_{i_k}\|_u = \|\lambda_k u_k\|$ for some $\lambda_k \in \mathbb{K}$ and

$$e_{i_k} \perp [v_1, \dots, v_{m_X}, e_{i_{m_X+1}}, \dots, e_{i_{k-1}}].$$

Next set $f(u_k) := e_{i_k}$ for $k = m_X+1, \dots, m_Y$. Define $f : j(X) + F_Y \rightarrow F$. Clearly f is an isometry and $f(j(X) + F_Y) \subset E_h^0$.

If \mathbb{K} is spherically complete, we are done, as $\{u_1, \dots, u_{m_Y}\}$ is an orthogonal base of Y by [12, Lemma 5.5 and Theorem 5.15]; hence, f is a required isometry defined on Y . If \mathbb{K} is non-spherically complete and $j(X) + F_Y \neq Y$, then, by [7, Proposition 2.1], Y is an immediate extension of $j(X) + F_Y$. Now, using Lemma 4.2, we extend f to the isometry defined on Y . \square

The last part of the proof of Proposition 4.6 uses Lemma 4.2 for the spherical completeness of the considered space F . In fact it is enough to assume that F contains a spherical completion of its every finite-dimensional linear subspace. This observation suggests another construction.

For each $g \in G$ set $I_g = \{i_{g,1}, i_{g,2}, \dots\}$ (note that I_g is countable). For every $n \in \mathbb{N}$ set $F_g^n := [e_{i_{g,1}}, \dots, e_{i_{g,n}}]$, a finite-dimensional linear subspace of $c_0(I_u)$ spanned by appropriate unit vectors. By $\widehat{F_g^n}$ denote a spherical completion of F_g^n such that for fixed $g \in G$ we have $F_g^n \subset \widehat{F_g^n} \subset \widehat{c_0(I_g)}$ and $\widehat{F_g^{n-1}} \subset \widehat{F_g^n}$ if $n > 1$.

Next, for every $g \in G$ define $F_g := \bigcup_n \widehat{F_g^n}$. Let $E_h := \bigoplus_{g \in G} \overline{F_g}$.

We are in a position to prove Theorem 4.7.

Theorem 4.7. *The space E_h is of universal disposition for the class \mathcal{U}_{FNA} .*

Proof. Let Y be a finite-dimensional non-archimedean normed space, let $X \subset Y$, and let $j : X \rightarrow E_h$ be an isometric embedding. We prove that there exists an isometry $f : Y \rightarrow E_h$ such that $\|j - f|_X\| < 1$. Then if we apply Theorem 1.3, the proof will be finished.

Set $t \in (0, 1)$ and choose a t -orthogonal base $\{x_1, \dots, x_{n_X}\}$ of X . Let $z_i := j(x_i)$, $i = 1, \dots, n_X$. Then for each $i \in \{1, \dots, n_X\}$ there exists a finite $G_i \subset G$, say $G_i = \{g_1, \dots, g_m\}$, and

$$z'_i \in \bigoplus_{g \in G_i} \overline{F_g}$$

for which

$$\|z'_i - z_i\| < \frac{t}{2} \|z_i\|.$$

Fix $i \in \{1, \dots, n_X\}$. Then we can write $z'_i = z'_{i,1} + \dots + z'_{i,m}$, where $z'_{i,k} \in \overline{F_{g_k}}$, $k = 1, \dots, m$. But, then we can select n_i and $w'_{i,1}, \dots, w'_{i,m}$ such that $w'_{i,k} \in \widehat{F_{g_k}^{n_i}}$, and $\|w'_{i,k} - z'_{i,k}\| < \frac{t}{2} \|z'_{i,k}\|$, $k = 1, \dots, m$. Denote $w_i := w'_{i,1} + \dots + w'_{i,m}$. Then

$$\begin{aligned} \|w_i - z_i\| &= \|w_i - z'_i + z'_i - z_i\| \leq \max\{\|w_i - z'_i\|, \|z'_i - z_i\|\} \\ &\leq \max\left\{\max_{k=1, \dots, m} \|w'_{i,k} - z'_{i,k}\|, \|z'_i - z_i\|\right\} < \frac{t}{2} \|z_i\|. \end{aligned}$$

Hence, by Lemma 4.1, $\{w_i : i \in \{1, \dots, n_X\}\}$ is a t -orthogonal set in $\bigoplus_{g \in G_0} \widehat{F_g^{n_0}}$, where $G_0 = G_1 \cup \dots \cup G_{n_X}$ and $n_0 = \max\{n_i : i \in \{1, \dots, n_X\}\}$.

Define the map $f : X \rightarrow \bigoplus_{g \in G_0} \widehat{F_g^{n_0}} \subset E_h$ setting $f(x_i) := w_i$, $i = 1, \dots, n_X$. Then for all $\lambda_i \in \mathbb{K}$ ($i = 1, \dots, n_X$) we have

$$f : \sum_{i=1}^{n_X} \lambda_i x_i \mapsto \sum_{i=1}^{n_X} \lambda_i w_i.$$

Consequently

$$\begin{aligned} \left\| f\left(\sum_{i=1}^{n_X} \lambda_i x_i\right) \right\| &= \left\| \sum_{i=1}^{n_X} \lambda_i w_i - \sum_{i=1}^{n_X} \lambda_i z_i + \sum_{i=1}^{n_X} \lambda_i z_i \right\| = \left\| \sum_{i=1}^{n_X} \lambda_i (w_i - z_i) + \sum_{i=1}^{n_X} \lambda_i z_i \right\| \\ &= \left\| \sum_{i=1}^{n_X} \lambda_i z_i \right\| = \left\| \sum_{i=1}^{n_X} \lambda_i j(x_i) \right\| = \left\| \sum_{i=1}^{n_X} \lambda_i x_i \right\| \end{aligned}$$

since

$$\left\| \sum_{i=1}^{n_X} \lambda_i (w_i - z_i) \right\| < \frac{t}{2} \max_{i=1, \dots, n_X} \|\lambda_i z_i\|$$

and, as $\{z_1, \dots, z_{n_X}\}$ is t -orthogonal, we have

$$\left\| \sum_{i=1}^{n_X} \lambda_i z_i \right\| \geq t \cdot \max_{i=1, \dots, n_X} \|\lambda_i z_i\|.$$

Thus f is isometric. Observe that

$$\begin{aligned} \|f - j\| &= \sup_{x \in X} \frac{\|(f - j)(x)\|}{\|x\|} = \sup_{\lambda_i \in \mathbb{K} \ (i=1, \dots, n_X)} \frac{\|\sum_{i=1}^{n_X} \lambda_i (f - j)(x_i)\|}{\|\sum_{i=1}^{n_X} \lambda_i x_i\|} \\ &\leq \sup_{\lambda_i \in \mathbb{K} \ (i=1, \dots, n_X)} \frac{\max_{i=1, \dots, n_X} \|\lambda_i (w_i - z_i)\|}{t \cdot \max_{i=1, \dots, n_X} \|\lambda_i x_i\|} \\ &\leq \sup_{\lambda_i \in \mathbb{K} \ (i=1, \dots, n_X)} \frac{\frac{t}{2} \cdot \max_{i=1, \dots, n_X} \|\lambda_i x_i\|}{t \cdot \max_{i=1, \dots, n_X} \|\lambda_i x_i\|} < 1. \end{aligned}$$

Now, we extend f on Y . We argue similarly as in the proof of the previous theorem. Choose $\{u_1, \dots, u_{m_X}, \dots, u_{m_Y}\}$, a maximal orthogonal set in Y such that $\{u_1, \dots, u_{m_X}\}$ is a maximal orthogonal set in X . By Lemma 4.3 we get $F_Y \perp j(X)$, where $F_Y = [u_{m_X+1}, \dots, u_{m_Y}]$.

Denote $v_k := f(u_k)$ for $k = 1, \dots, m_X$, and for each $k \in \{m_X+1, \dots, m_Y\}$ choose $g_k \in G$ and $i_{g_k, n_k} \in I_{g_k}$, denoting for simplicity $j_k := i_{g_k, n_k}$, such that $\|e_{j_k}\|_u = \|\lambda_k u_k\|$ (for some $\lambda_k \in \mathbb{K}$) and

$$e_{j_k} \perp [v_1, \dots, v_{m_X}, e_{i_{m_X+1}}, \dots, e_{j_{k-1}}].$$

Then, setting $f(u_k) := e_{j_k}$, where $k = m_X+1, \dots, m_Y$, we extend f on $X + F_Y$. Let $p_0 = \max\{n_k : k = m_X+1, \dots, m_Y\}$ and $G_1 = \{g_k : k = m_X+1, \dots, m_Y\}$. The map f is an isometry and $f(X + F_Y) \subset H$, where

$$H = \bigoplus_{g \in G_0 \cup G_1} \widehat{F_g^{\max\{n_0, p_0\}}}.$$

If \mathbb{K} is spherically complete, the proof is completed, since $\{u_1, \dots, u_{n_Y}\}$ is an orthogonal base of Y by [12, Lemma 5.5 and Theorem 5.15]. This shows that f is a required isometry defined on Y . Assume that \mathbb{K} is non-spherically complete. Since Y is an immediate extension of $X + F_Y$ and H is spherically complete as a finite direct sum of spherically complete spaces (see [12, 4A]), we apply Lemma 4.2 to extend f to the isometry defined on Y . Since $\|j - f|_X\| < 1$, as we proved above, we apply Theorem 1.3. The proof is finished. \square

This yields the following interesting

Corollary 4.8. (1) *If \mathbb{K} is spherically complete and $(0, \infty)$ is the union of at most countably many cosets of $|\mathbb{K}^*|$, the isometrically universal space E_h for the class of non-archimedean Banach spaces of countable type is also of universal disposition for the class \mathcal{U}_{FNA} .*

(2) *There exist non-archimedean Banach spaces of universal disposition for the class \mathcal{U}_{FNA} which are not isometrically isomorphic.*

Proof. Recall that if \mathbb{K} is spherically complete, then every finite-dimensional normed space over \mathbb{K} is spherically complete (see [12, Theorem 4.2 and Corollary 4.6]). Thus, $E_h = (c_0(I_u), \|\cdot\|_u) = E_u$. The remaining part of the proof of (1) follows from Proposition 4.4. To prove (2) consider the spaces E_h and $\widehat{E_u}$ (by Theorems 4.6 and 4.7 both are spaces of universal disposition for the class \mathcal{U}_{FNA}) assuming that \mathbb{K} is spherically complete. Then $E_h = E_u$. Since $\|E_u^*\|_u$ is a dense subset of $(0, \infty)$, we can choose $\{i_1, i_2, \dots\} \subset I_u$ such that $1 \geq \|e_{i_1}\|_u > \|e_{i_2}\|_u > \dots > \frac{1}{2}$ (where e_{i_k} are unit vectors of E_u). Set

$x_n := \sum_{k=1}^n e_{i_k}$ ($n \in \mathbb{N}$). Then, the balls $V_n := \{x \in E_u : \|x - x_n\|_u \leq \|e_{i_{n+1}}\|_u\}$ form a shrinking sequence in E_u . But $\bigcap_{n \in \mathbb{N}} V_n = \emptyset$; hence, E_u is not spherically complete. Clearly $\widehat{E_u}$, as a spherical completion of E_u , is spherically complete, thus we imply that $E_h (=E_u)$ and $\widehat{E_u}$ are not isometrically isomorphic. \square

Remark 4.9. Note that (see Remark 4.5) if \mathbb{K} is non-spherically complete then the space ℓ^∞ is not of universal disposition for the class \mathcal{U}_{FNA} . Indeed, take $Y = \mathbb{K}_v^2$, $e_1 = (1, 0, 0, \dots) \in \ell^\infty$ and define the isometric embedding $i : [e_1] \rightarrow \mathbb{K}_v^2 : e_1 \mapsto (1, 0)$. On the other hand, by [8, Proposition 3.2], every two-dimensional linear subspace of ℓ^∞ containing e_1 has two non-zero orthogonal elements. Thus, there is no isometric embedding $f : Y \rightarrow \ell^\infty$ such that $f(1, 0) = e_1$.

REFERENCES

- [1] A. Avilés, F. Cabello Sánchez, J. M. F. Castillo, M. González, Y. Moreno, *Banach space of universal disposition*, J. Funct. Anal. **261** (2011), 2347—2361.
- [2] J. Garbulińska, W. Kubiś, *Remarks on Gurariĭ spaces*, Extracta Math. **26** (2011), 235—269.
- [3] V. I. Gurariĭ, *Spaces of universal placement, isotropic spaces and a problem of Mazur on rotations of Banach spaces*, Sibirsk. Mat. Z. **7** (1966), 1002—1013 (in Russian).
- [4] W. Kubiś, *Game-theoretic characterization of the Gurariĭ space*. preprint
- [5] W. Kubiś, S. Solecki, *Proof of uniqueness of the Gurariĭ space*. Israel J. Math. **195** (2013), 449—456.
- [6] A. Kubzdela, *On finite-dimensional normed spaces over \mathbb{C}_p* . Contemp. Math. **384** (2005), 169—185.
- [7] A. Kubzdela, *On non-Archimedean Hilbertian spaces*. Indag. Math. New Ser. **19** (2008), 601—610.
- [8] A. Kubzdela, *On some geometrical properties of linear subspaces of ℓ^∞* , Contemp. Math. **551** (2011), 157—161.
- [9] A. J. Lazar, J. Lindenstrauss, *Banach spaces whose duals are L_1 spaces and their representing matrices*, Acta Math. **126** (1971), 165—193.
- [10] W. Lusky *The Gurariĭ spaces are unique*, Arch. Math. (Basel) **27** (1976), 627—635.
- [11] C. Perez-Garcia, W. H. Schikhof, *Locally Convex Spaces over Non-archimedean Valued Fields*. Cambridge University Press, Cambridge, 2010.
- [12] A. C. M. van Rooij, *Non-Archimedean Functional Analysis*. Marcel Dekker, New York, 1978.
- [13] W. H. Schikhof, *Ultrametric Calculus*. Cambridge University Press, 1984.
- [14] W. H. Schikhof, *The complementation property of ℓ^∞ in p -adic Banach spaces*. In: *p-adic Analysis*, edited by F. Baldassarri, S. Bosh and B. Dworak. Lecture Notes in Mathematics, Springer-Verlag, Berlin (1990), 342—350.

ADAM MICKIEWICZ UNIVERSITY, POZNAŃ, POLAND AND INSTITUTE OF MATHEMATICS, CZECH ACADEMY OF SCIENCES, CZECH REPUBLIC

INSTITUTE OF MATHEMATICS, CZECH ACADEMY OF SCIENCES, CZECH REPUBLIC AND CARDINAL STEFAN WYSZYŃSKI UNIVERSITY, WARSAW, POLAND

INSTITUTE OF CIVIL ENGINEERING, UNIVERSITY OF TECHNOLOGY, POZNAŃ, POLAND